Then the constant value U_0 will be maintained so long as V is maintained. Finally when v returns to 0 the coefficient $M+W-AL$ may again be regarded as constant and it is seen that u comes to the value

$$U_1 = -\frac{AL}{M+W_0-AL}V + \frac{AL}{M+W_1-AL}V, \text{ where } W_1 \text{ is the final } W.$$

This value of u is not 0 but positive, and indeed is necessarily so in order to counterbalance the negative momentum of the water which has escaped. The figure represents the variation of v and $-u$.

Other reasonable assumptions might be made, for instance Torricelli's law of efflux. Moreover the actual displacement of the tank could be obtained by integrating u. But sufficient has been said to show how readily the problem may be treated as far as its purely mechanical side is concerned. As it is none too easy to find really good and yet essentially different problems which require for their solution merely the fundamental principles of momentum, energy, and moment of momentum, it may not be amiss to point out that the above problem may be relieved of its hydrodynamical difficulties but otherwise preserved intact by considering a small number of beads on a wire lying in a vertical plane and consisting of a series of segments at different inclinations to the horizontal.

THE THEORY OF INVERSION AND THE QUADRATIC RECIPROCAL TRANSFORMATION.

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1. The following discussion exhibits the theory of inversion in its proper light as a quadratic reciprocal transformation, and makes clear why inversion throws circles into circles but generally a curve of degree n into a curve of degree $2n$.

2. Consider a conic K and a point M. A correspondence between the points of the plane may now be set up as follows: To any point P make correspond the point P' of intersection of the line MP with the polar of P with respect to K. The point P' goes by this process into the point P again by the fundamental theorem in the theory of poles and polars: If the polar of P passes through P', the polar of P' passes through P.

If the point P moves along a straight line the point P' moves along a conic.

For P' is the locus of intersection of two projective pencils with centers at M and at the pole of the line upon which P moves. This conic thus

passes through a point M. A slight consideration will show also that it passes through the points A and B where the tangents from M to K touch K. For let P be on one of these tangents, then the line PM meets the polar of P in one of the points of tangency. Further, if P lies on the line AB the corresponding point P' is at M. More generally, we shall prove that

The transformation above described tranforms a curve of degree n into a curve of degree $2n$ which passes n times through the points A, B, and M.

For let C be a curve of degree n. It transforms into a curve C' of degree which it is proposed to determine. Cut across C' by a straight line a. Transform now again and C' goes back into C and the line a into a conic a'. The $2n$ points of intersection of a' with C correspond to the points of intersection of the line a and C'. Further, the curve C cuts the line AB in n points. The curve C' then passes n times through M. Similarly for points A and B.

The transformation is a little different from the ordinary transformation by which lines go into conics in that ordinarily points on a side of the singular triangle go into the *opposite* vertex. Here, while this is the case for the side AB, it is not the case for the other two sides, MA and MB.

The usual considerations apply for curves that go through one or more of the vertices A, B, and M of the singular triangle. Thus a conic generally goes into a quartic with three double points A, B, and M. If the conic goes through M the quartic degenerates into a straight line AB and a cubic through A, B and M with double points at M. If the conic goes through the points A and B the quartic becomes a conic and two straight lines. The two lines are MA and MB, the conic goes through A and B. Finally, if the conic goes through M, A, and B it transforms into a quartic made up of four straight lines one of which is the line that corresponds to that conic by the transformation; the other three being the sides of the singular triangle.

Consider now the special case where K is a circle and M is the center of that circle. Everything noted above still holds. If P moves along a line the point P' moves along a conic through M, A, and B. It is now seen that the points A and B are the points, usually denoted by I and J, where the circle K meets the polar of M, that is the line at infinity. If we define a circle as a conic through I and J, we have a transformation that throws a straight line into a circle. Moreover, a circle by this transformation goes into a circle, for as we have seen a conic through A and B goes into a conic through A and B. A conic generally goes into a quartic with double point at the center of K and double points at the circular points at infinity.

The metrical properties of the theory of inversion are now easily built up. It is not necessary to indicate the process further.